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Thermodynamic Formalism for Contracting Lorenz Flows

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Abstract We study the expansion properties of the contracting Lorenz flow introduced by Rovella via thermodynamic formalism. Specifically, we prove the existence of an equilibrium state for the natural potential $\hat{\varphi}_t(x, y, z) := -t \log J^{cu}_{(x,y,z)}$ for the contracting Lorenz flow and for t in an interval containing [0, 1]. We also analyse the Lyapunov spectrum of the flow in terms of the pressure.

Keywords Singular-hyperbolic attractor · Lorenz-like flow · Thermodynamic formalism · Lyapunov exponents · Multifractal spectra

1 Introduction

The Lorenz flow [28] is one of the key examples in the theory of dynamical systems due to the chaotic nature of its dynamics, its robustness and its connection with hydrodynamical systems. The Lorenz attractor, a 'strange attractor' with a characteristic butterfly shape, has extremely rich dynamical properties which have been studied from a variety of viewpoints:

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topological, geometric and statistical, see [4, 44]. Part of the reason for the richness of the Lorenz flow is the fact that it has an equilibrium, i.e., a fixed point, which is accumulated by regular orbits (orbits through points where the corresponding vector field does not vanish) which prevents the flow from being uniformly hyperbolic. Indeed it is one of the motivating examples in the study of non-uniformly hyperbolic dynamical systems [31]. It is also robust in the sense that nearby flows also possess strange attractors with similar properties. The Lorenz equations can be studied using geometric models of the Lorenz flow, see [2, 22]. It was shown by Tucker [45] that the Lorenz equations do indeed support a geometric Lorenz flow.

The classical geometric Lorenz flow is expanding. This corresponds to the Lyapunov exponents at the origin, λ_s and λ_u, the stable and unstable exponents respectively, having $\lambda_u + \lambda_s > 0$. A Rovella-like attractor [41] is the maximal invariant set of a geometric flow whose construction is very similar to the one that gives the geometric Lorenz attractor, [2, 4, 22], except for the fact that the eigenvalue relation $\lambda_u + \lambda_s > 0$ there is replaced by $\lambda_u + \lambda_s < 0$. As in the case for the geometric Lorenz attractor, a Rovella attractor has a global cross section: a line $\Gamma \subset \mathbb{R}^3$, and a first return map $\tilde{f}$ defined on $\mathbb{R}^3 \setminus \Gamma$ that preserves a one-dimensional foliation which is contracted under the action of $\tilde{f}$. Thus, as in the case of the geometrical model for the Lorenz flow, it is possible to study the dynamics of a Rovella flow through a 1-dimensional map obtained quotienting though the leaves of this contracting foliation. Unlike the one-dimensional Lorenz map obtained from the usual construction of the geometric Lorenz attractor, a one-dimensional Rovella map has a criticality at the origin, caused by the eigenvalue relation $\lambda_u + \lambda_s < 0$. In Fig. 3 below we present some possible “Rovella one-dimensional maps” obtained through quotienting out the stable direction of the return map to the global cross-section of the attractor. In Sect. 3 we explain this procedure.

In this paper we will study the geometric model of the Rovella-like attractor from the point of view of thermodynamic formalism. This theory studies the multifractal properties of the system (see [35]), providing precise characterisations of the dimension theory, as well as giving insight into the statistical properties of the system. In the study of thermodynamic formalism, one takes a dynamical system $(f_s)_s : X \to X$ and a relevant potential $\varphi : X \to \mathbb{R}$ and studies the statistical properties of the system through the properties of the pressure and the equilibrium states of the triple $(X, (f_s)_s, \varphi)$. This theory was developed for hyperbolic dynamical systems by Sinai, Ruelle and Bowen [10, 42, 43] in the context of Hölder potentials on hyperbolic dynamical systems, and has mainly been applied to Axiom A systems and Anosov diffeomorphisms, see e.g. [6, 26].

The potentials which tell us most about the system involve the Jacobean of the flow/map. For discrete smooth conformal systems one would consider $\varphi = \log |Df|$. Knowledge of the pressure and equilibrium states with respect to the family $t\varphi$, the family of ‘natural/geometric’ potentials, give us very fine information on the expansion properties of the system. This is the Lyapunov spectrum. For discrete uniformly hyperbolic systems the theory of thermodynamic formalism is already fairly well developed, see for example [33, 46]. However, for discrete conformal non-uniformly hyperbolic dynamical systems the theory is currently seeing a lot of activity, for example [12, 13, 20, 21, 24, 25, 32, 38, 40]. In the case of flows, thermodynamic formalism has been studied in the hyperbolic case in [8–10, 14, 36, 48]. In the non-uniformly hyperbolic case, the main contribution was made by Barreira and Iommi [7] who considered thermodynamic formalism for suspension flows over countable Markov shifts.

To understand the Jacobean for the Lorenz flow we note that the tangent space can be split into three directions: the flow direction, which has neutral expansion, the expanding/unstable direction and the contracting/stable direction. The study of the Lyapunov spectrum in the Rovella flow case is particularly complicated since, in contrast to the expanding Lorenz case, we have to deal with points where a derivative is zero.

The interesting part of the dynamics is in the expanding part of the attractor, so we consider the Jacobean restricted to the expanding direction. This situation can be modelled by a suspension flow over a countable Markov shift as in [7], but our approach uses a simpler suspension flow allied to the results of Iommi and Todd [24, 25]. (Note that in [24, 25] a countable Markov shift was used to produce the equilibrium states and information on the Lyapunov spectra.) Our analysis captures the points which are typical for the physical measure as well as for many other points captured by nearby measures. As mentioned above, we use the common approach (see [5, 19, 23, 29–31]) of analysing Lorenz-like flows by taking Poincaré sections in such a way that we obtain a one-dimensional map. Note that our results hold for a larger class of maps than just the Rovella type of Lorenz flow. We consider flows which have a Poincaré section with the dynamics of maps considered in the appendix of [24].

2 The Main Results

As sketched in the introduction and explained in greater detail below, given a flow $\hat{f} = (\hat{f}_s)_s \subset \hat{\mathcal{F}}$ and some $t \in \mathbb{R}$, we prove the existence of an equilibrium state for the potential $\hat{\varphi}_t(x, y, z) := -t \log J^{cu}_{(x,y,z)}$ where $J^{cu}_{(x,y,z)}$ is the Jacobean of $\hat{f}$ at (x, y, z) in the unstable direction. Here $\hat{\mathcal{F}}$ includes the contracting Lorenz flow introduced in [41]. The set of potentials $\{\hat{\varphi}_t(x, y, z) : t \in \mathbb{R}\}$ is the natural class to consider for these maps. Indeed, analysis of these potentials also allows us to express the Lyapunov spectrum of the flow in terms of the pressure. Recall that a *contracting Lorenz flow* $\hat{f}$ is a flow with a unique singularity at the origin 0, defined in a compact neighbourhood $\mathcal{C}$ of 0 satisfying the following properties:

(1) the restriction of the flow to a small neighbourhood $\mathcal{Q} \subset \overline{\mathcal{Q}} \subset \mathcal{C}$ is a linear flow L with a unique singularity at 0,
(2) the eigenvalues λ_i, $1 \leq i \leq 3$, of $DL(0)$ are all real and satisfy $-\lambda_2 > -\lambda_3 > \lambda_1 > 0$.

It was proved in [41] that, under certain additional conditions, the maximal positive $\hat{f}$-invariant set $\Lambda \subset \mathcal{Q}$ is a transitive attractor.

In order to give our main results for these systems we first need to introduce some basic notions from thermodynamic formalism. For references on the general theory, see for example [10, 14, 26, 35].

2.1 Thermodynamic Formalism

We begin by giving definitions for discrete time dynamical systems $f : X \to X$, and will then generalise to the flow case. We let

$$\mathcal{M} = \mathcal{M}(f) := \left\{\text{measures } \mu : \mu \circ f^{-1} = \mu \text{ and } \mu(X) = 1\right\}.$$

Given a *potential* $\varphi : X \to [-\infty, \infty]$, the *pressure* of φ with respect to f is defined as

$$P(\varphi) = P(f, \varphi) := \sup\left\{h(\mu) + \int \varphi \, d\mu : \mu \in \mathcal{M} \text{ and } -\int \varphi \, d\mu < \infty\right\},$$

where $h(\mu)$ denotes the measure theoretic entropy of f with respect to μ. As in [26], the quantity $h(\mu) + \int \varphi \, d\mu$ is referred to as the *free energy* of μ with respect to (X, f, φ). A measure $\mu \in \mathcal{M}$ maximising the free energy, i.e., with $h(\mu) + \int \varphi \, d\mu = P(\varphi)$, is called an *equilibrium state*.

Similarly for a flow $\hat{f}$, we define the set of $\hat{f}$-invariant measures as

$$\mathcal{M} = \mathcal{M}(\hat{f}) := \left\{ \text{measures } \hat{\mu} : \hat{\mu}(\hat{f}_s^{-1}(A)) = \hat{\mu}(A) \text{ for all } s \geq 0 \text{ and } \hat{\mu}(\hat{X}) = 1 \right\}.$$

Moreover, for a potential $\hat{\varphi} : \hat{X} \to \mathbb{R}$, the pressure of $(\hat{X}, \hat{f}, \hat{\varphi})$ is defined as

$$P(\hat{f}, \hat{\varphi}) := \sup \left\{ h(\hat{f}, \hat{\mu}) + \int \hat{\varphi} \, d\hat{\mu} : \hat{\mu} \in \mathcal{M}(\hat{f}) \text{ and } - \int \hat{\varphi} \, d\hat{\mu} < \infty \right\}.$$

(For more details of the entropy of flows, see Sect. 5, in particular (17).)

For a flow $(\hat{f}_s)_s : \hat{X} \to \hat{X}$ in our class $\hat{\mathcal{F}}$, as in [5, 30, 31] at each point $(x, y, z) \in \mathbb{R}^3$, the tangent space for the flow $\hat{f}$ has a splitting $E_x^{cu} \oplus E_x^s$ where E_x^s is tangent to the stable direction and E_x^{cu} is tangent to the centre unstable direction (see Sect. 3 for more details). We are interested in the potential

$$\hat{\varphi}_t(x, y, z) := -t \log J_{(x,y,z)}^{cu} \tag{1}$$

which is the Jacobean of the differential in the centre unstable direction at the point (x, y, z). This potential gives rise to a natural class of equilibrium states, which can be seen as selecting out the sets in $\mathbb{R}^3$ with different rates of asymptotic expansion by the flow (see also Theorem B). For the following theorem, our first main result for contracting Lorenz flows, we consider this potential for $t \in (t^-, t^+)$. The values of $t^- \leq 0$ and $t^+ \geq 1$ are given below in (14).

Theorem A *Let $\hat{f} \in \hat{\mathcal{F}}$. Then for all $t \in (t^-, t^+)$, there is an equilibrium state $\hat{\mu}_t$ for $\hat{\varphi}_t(x, y, z)$.*

The set $\hat{I} \subset \mathbb{R}^3$, defined more precisely later, is the set on which the flow $\hat{f}$ is defined. Given a potential $\hat{\varphi} : \hat{I} \to \mathbb{R}$, and $\alpha \in \mathbb{R}$, let

$$K^{\hat{\varphi}}(\alpha) := \left\{ (x, y, z) \in \hat{I} : \lim_{u \to \infty} \frac{1}{u} \int_0^u \hat{\varphi}(\hat{f}_s(x, y, z)) \, ds = \alpha \right\}$$

and

$$K^{\hat{\varphi}'} := \left\{ (x, y, z) \in \hat{I} : \lim_{u \to \infty} \frac{1}{u} \int_0^u \hat{\varphi}(\hat{f}_s(x, y, z)) \, ds \text{ does not exist} \right\}.$$

In our second main theorem for contracting Lorenz flows, we take $K(\alpha) := K^{\log J^{cu}}(\alpha)$. The *Lyapunov spectrum* of $(\hat{I}, \hat{f})$ is the map

$$\alpha \mapsto \mathfrak{L}_{\hat{f}}(\alpha) := \dim_H(K(\alpha)),$$

where $\dim_H$ denotes the Hausdorff dimension of a set.

In our analysis of $\mathfrak{L}_{\hat{f}}$, we will use the potentials $\hat{\varphi}_t$, for certain parameters $t \in \mathbb{R}$, and their equilibrium states. The flows we consider and the potentials $\hat{\varphi}_t$ have a natural relation with piecewise C^2 maps f on an interval I and the natural potentials

$$\varphi_t(x) := -t \log |Df(x)|. \tag{2}$$

Often it can be shown that an equilibrium state for one such potential φ_1 is an absolutely continuous invariant probability measure (acip) μ_{ac}. This is also an SRB measure.

Defining, for a measure $\mu \in \mathcal{M}$, the *Lyapunov exponent* of (I, f, μ) by

$$\lambda(\mu) := \int \log |Df| \, d\mu,$$

any equilibrium state μ_t for φ_t therefore satisfies

$$h(\mu_t) - t\lambda(\mu_t) = P(\varphi_t).$$

We also define the pressure function:

$$p(t) := P(-t \log |Df|).$$

In the following theorem, we give a relation between Lyapunov spectrum and the pressure function on a certain domain $(\alpha_1, \alpha_2) \subset \mathbb{R}$ which is defined later in (15). Note that in general the interval $(\alpha_1, \alpha_2]$ contains the Lyapunov exponents of both the SRB measure and the measure of maximal entropy. We restrict our analysis to a subset of maps $\hat{\mathcal{F}}_{ac} \subset \hat{\mathcal{F}}$, which will be defined below.

Theorem B *Let $\hat{f} \in \hat{\mathcal{F}}_{ac}$. Then for all $\alpha \in (\alpha_1, \alpha_2)$, the Lyapunov spectrum satisfies the following relation*

$$\mathfrak{L}_{\hat{f}}(\alpha) - 2 = \frac{1}{\alpha} \inf_{t \in \mathbb{R}} (p(t) + t\alpha) = \frac{1}{\alpha} (p(t) + t_\alpha \alpha),$$

where t_α is such that $Dp(t_\alpha) = -\alpha$.

Remark 1 Note that it can be shown that for potentials φ_t and $\hat{\varphi}_t$, with corresponding equilibrium states μ_t and $\hat{\mu}_t$, we have $\mathfrak{L}_{\hat{f}}(\alpha) - 2 = \frac{h(\mu_{t_\alpha})}{\alpha} = \frac{h(\hat{\mu}_{t_\alpha})}{\alpha}$ for α and t_α as in Theorem B. Moreover, $\hat{\mu}_{t_\alpha}(\mathbb{R}^3 \setminus K(\alpha)) = 0$.

We will prove Theorems A and B by first reducing the study of maps in $\hat{\mathcal{F}}$ to a class of maps on the unit square and then reducing further to a class of maps on the unit interval. This is explained in the following two sections.

3 Construction of a Rovella Flow

In this section we will consider a class of three dimensional flows which will be defined axiomatically. To show that these axioms are verified in the geometric contracting Lorenz models we give a detailed construction of this model. This construction can also be found in for example [4, 19, 41].

We first analyse the dynamics in a neighbourhood of the singularity at the origin, and then we complete the flow, imitating the butterfly shape of the original Lorenz flow.

We start with a linear system $(\dot{x}, \dot{y}, \dot{z}) = (\lambda_1 x, \lambda_2 y, \lambda_3 z)$, with λ_i, $1 \le i \le 3$ satisfying the relations

$$-\lambda_2 > -\lambda_3 > \lambda_1 > 0, \quad \beta > \ell + 3, \quad \beta := -\frac{\lambda_2}{\lambda_1}, \ \ell := -\frac{\lambda_3}{\lambda_1}. \tag{3}$$

This vector field will be considered in the cube $[-1,1]^3$. For this linear flow, the trajectories are given by

$$\hat{f}_s(x_0, y_0, z_0) = (x_0 e^{\lambda_1 s}, y_0 e^{\lambda_2 s}, z_0 e^{\lambda_3 s}), \tag{4}$$

where $(x_0, y_0, z_0) \in \mathbb{R}^3$ is an arbitrary initial point near $p = (0,0,0)$.

Now let $\Sigma := \{(x, y, 1) : |x| \leq 1/2,\ |y| \leq 1/2\}$ and consider

$$\Sigma^- := \big\{(x, y, 1) \in \Sigma : x < 0\big\}, \qquad \Sigma^+ := \big\{(x, y, 1) \in \Sigma : x > 0\big\} \quad \text{and}$$
$$\Sigma^* := \Sigma^- \cup \Sigma^+ = \Sigma \setminus \Gamma, \quad \text{where } \Gamma := \big\{(x, y, 1) \in \tilde{I} : x = 0\big\}.$$

Σ is a transverse section to the linear flow and every trajectory crosses Σ in the direction of the negative z axis.

Consider also $\tilde{\Sigma} := \{(x, y, z) : |x| = 1\} = \tilde{\Sigma}^- \cup \tilde{\Sigma}^+$ with $\tilde{\Sigma}^\pm := \{(x, y, z) : x = \pm 1\}$. For each $(x_0, y_0, 1) \in \Sigma^*$ the next time $s > 0$ such that $\hat{f}_s(x_0, y_0, 1) \in \tilde{\Sigma}$ is given by

$$s(x_0) := -\frac{1}{\lambda_1} \log |x_0| \tag{5}$$

which depends on $x_0 \in \tilde{\Sigma}^*$ only and is such that $s(x_0) \to +\infty$ when $x_0 \to 0$.

Hence, using (5), we get (where $\operatorname{sgn}(x) := x/|x|$ for $x \neq 0$)

$$\hat{f}_{s(x_0)}(x_0, y_0, 1) = \big(\operatorname{sgn}(x_0), y_0 e^{\lambda_2 \cdot s(x_0)}, e^{\lambda_3 \cdot s(x_0)}\big) = \big(\operatorname{sgn}(x_0), y_0 |x_0|^\beta, |x_0|^\ell\big).$$

Consider $L : \Sigma^* \to \tilde{\Sigma}^\pm$ defined by

$$L(x, y, 1) := \big(\operatorname{sgn}(x), y|x|^\beta, |x|^\ell\big). \tag{6}$$

Clearly each segment $\Sigma^* \cap \{x = x_0\}$ is taken by L to another segment $\tilde{\Sigma}^\pm \cap \{z = z_0\}$ as sketched in Fig. 1.

It is easy to see that $L(\Sigma^\pm)$ has the shape of a cusp triangle with a vertex $(\pm 1, 0, 0)$, a cusp point at the boundary of the triangle. Since L is a linear flow, it preserves the vertical

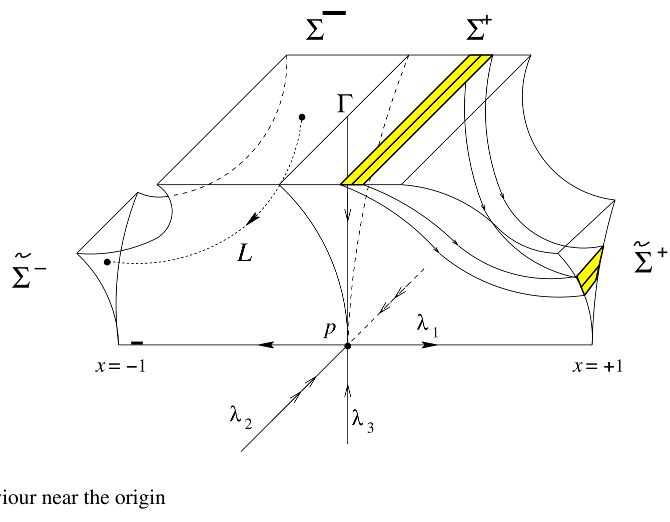

Fig. 1 Behaviour near the origin

foliation $\mathcal{F}^s$ of Σ whose leaves are given by the lines $x = x_0$. We shall further assume that $L(\Sigma^{\pm})$ are uniformly compressed in the y-direction.

3.1 The Random Turns Around the Origin

To imitate the random turns of a regular orbit around the origin and obtain a butterfly shape for our flow, we proceed as follows.

Recall that the fixed point p at the origin is hyperbolic and so its stable $W^s(p)$ and unstable $W^u(p)$ manifolds are well defined, [34]. Observe that $W^u(p)$ has dimension one and so it has two branches, $W^{u,\pm}(p)$ and $W^u(p) = W^{u,+}(p) \cup \{p\} \cup W^{u,-}(p)$.

The sets $\tilde{\Sigma}^{\pm}$ should return to the cross section Σ through a flow described by a suitable composition of a rotation $R_{\pm}$, an expansion $E_{\pm\theta}$ and a translation $T_{\pm}$.

The rotation $R_{\pm}$ has axis parallel to the y-direction, which is orthogonal to the x-direction (which is parallel to the local branches $W^{u,\pm}(p)$). More precisely it is such that if $(x, y, z) \in \tilde{\Sigma}^{\pm}$, then

$$R_{\pm}(x, y, z) = \begin{pmatrix} 0 & 0 & \pm 1 \\ 0 & 1 & 0 \\ \pm 1 & 0 & 0 \end{pmatrix}. \tag{7}$$

The expansion occurs only along the x-direction, so the matrix of E_{θ} is given by

$$E_{\pm\rho}(x, y, z) = \begin{pmatrix} \rho & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{8}$$

with $\rho \cdot (\frac{1}{2})^{\ell} < 1$. This condition is to ensure that the image of the resulting map is contained in Σ.

The translation $T_{\pm} : \mathbb{R}^3 \to \mathbb{R}^3$ is chosen such that the unstable direction starting from the origin is sent to the boundary of Σ and the image of both $\tilde{\Sigma}^{\pm}$ are disjoint. These transformations $R_{\pm}, E_{\pm\rho}, T_{\pm}$ take line segments $\tilde{\Sigma}^{\pm} \cap \{z = z_0\}$ into line segments $\Sigma \cap \{x = x_1\}$, and so does the composition $T_{\pm} \circ E_{\pm\rho} \circ R_{\pm}$.

This composition of linear maps describes a vector field in a region outside $[-1, 1]^3$ in the sense that one can use the above matrices to define a vector field V such that the time one map of the associated flow realises $T_{\pm} \circ E_{\pm\rho} \circ R_{\pm}$ as a map $\tilde{\Sigma}^{\pm} \to \Sigma$. This will not be explicit here, since the choice of the vector field is not really important for our purposes.

The above construction allows us to describe, for each $s \in \mathbb{R}$, the orbit $f_s(x)$ of each point $x \in \tilde{I}$: the orbit will start following the linear field until $\tilde{\Sigma}^{\pm}$ and then it will follow V coming back to Σ and so on. Let us denote by

$$\mathcal{B} := \{\hat{f}_s(x) : x \in \Sigma, s \in \mathbb{R}^+\}$$

the set where this flow acts. The geometric contracting Lorenz flow is then the couple $(\mathcal{B}, \hat{f}_s)$ defined in this way.

The Poincaré first return map will thus be defined by $\tilde{f} : \Sigma^* \to \Sigma$ as

$$\tilde{f}(x, y) = \begin{cases} T_+ \circ E_{+\rho} \circ R_+ \circ L(x, y, 1) & \text{for } x > 0 \\ T_- \circ E_{-\rho} \circ R_- \circ L(x, y, 1) & \text{for } x < 0 \end{cases} \tag{9}$$

The combined effects of $T_{\pm} \circ R_{\pm}$ and L on lines implies that the foliation $\mathcal{F}^s$ of Σ given by the lines $\Sigma \cap \{x = x_0\}$ is invariant under the return map. In other words, we have

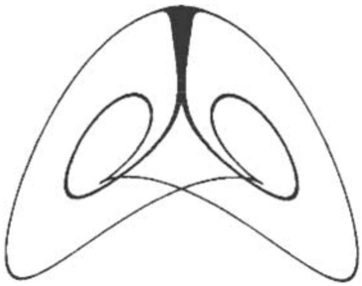

Fig. 2 A Rovella flow

($\star$) *for any given leaf γ of $\mathcal{F}^s$, its image $F(\gamma)$ is contained in a leaf of $\mathcal{F}^s$, and the condition $\beta > \ell + 3$ guarantees that $\mathcal{F}^s$ is a C^3-foliation.*

3.2 An Expression for the First Return Map

Let $I := [-1/2, 1/2]$. Combining equations (6) with the effect of the rotation composed with the expansion and the translation, we obtain that $\tilde{f}$ must have the form

$$\tilde{f}(x, y) = \big(f_{Ro}(x), g_{Ro}(x, y)\big)$$

where $f_{Ro} : I \setminus \{0\} \to I$ and $g_{Ro} : (I \setminus \{0\}) \times I \to I$ are given by

$$f_{Ro}(x) = \begin{cases} f_1(x^\ell) & \text{if } x < 0, \\ f_0(x^\ell) & \text{if } x > 0, \end{cases} \quad \text{with } f_i = (-1)^i \rho \cdot x + d_i, i \in \{0, 1\}, \quad \text{and}$$
$$g_{Ro}(x, y) = \begin{cases} g_1(x^\ell, y \cdot x^\beta) & \text{if } x < 0, \\ g_0(x^\ell, y \cdot x^\beta) & \text{if } x > 0, \end{cases} \tag{10}$$

where $g_1|I^- \times I \to I$ and $g_0|I^+ \times I \to I$ are suitable affine maps. Here $I^- = [-1/2, 0)$, $I^+ = (0, 1/2]$. Note that conditions (f1)–(f5) below determine the precise form of the constants d_i.

3.2.1 Properties of the Map g_{Ro}

Observe that by construction, g_{Ro} in (9) is piecewise C^3. Moreover, we have the following bounds on its partial derivatives:

(a) For all $(x, y) \in \Sigma^*, x > 0$, we have $\partial_y g_{Ro}(x, y) = x^\beta$. As $\beta > 1$, $|x| \le 1/2$, there is $0 < \lambda < 1$ such that

$$|\partial_y g_{Ro}| < \lambda. \tag{11}$$

The same bound works for $x < 0$.

(b) For all $(x, y) \in \Sigma^*, x \neq 0$, we have $\partial_x g_{Ro}(x, y) = \beta \cdot x^{\beta-\ell}$. As $\beta - \ell > 3$ and $|x| \le 1/2$, we get

$$|\partial_x g_{Ro}| < \infty. \tag{12}$$

Item (a) above implies that the map $\tilde{f} = (f_{Ro}, g_{Ro})$ is uniformly contracting on the leaves of the foliation $\mathcal{F}^s$: there is $C > 0$ such that

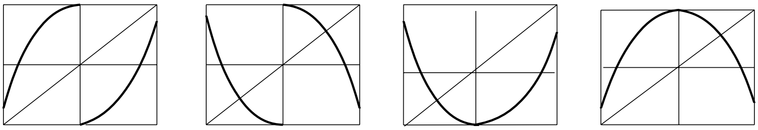

Fig. 3 Possible Rovella maps

($\star\star$) if γ is a leaf of $\mathcal{F}^s$ and $x, y \in \gamma$, then $\mathrm{dist}(\tilde{f}^n(x), \tilde{f}^n(y)) \le \lambda^n \cdot C \cdot \mathrm{dist}(x, y)$

where λ can be chosen as the one given by (11).

3.2.2 *Properties of the One-dimensional Map f_{Ro}*

Next we outline the main features of f_{Ro}.

The following properties are easily implied from the construction of $\hat{f}$:

(f1) By (10) and the way $T_\pm$ is defined, f_{Ro} is discontinuous at $x = 0$. The lateral limits $f_{Ro}(0^\pm)$ do exist, $f_{Ro}(0^\pm) = \pm\frac{1}{2}$,
(f2) f_{Ro} is C^3 on $I \setminus \{0\}$. As $\beta > \ell + 3$, we get $\lim_{x\to 0^+} Df_{Ro}(x) = 0 = \lim_{x\to 0^-} Df_{Ro}(x)$, and the order of f_{Ro} at $x = 0$ is $\ell - 1 > 0$.

By the convexity properties of f_{Ro} we then obtain that

$$Df_{Ro}(x) > 0 \quad \text{for all } x \in I \setminus \{0\}.$$

(f3) $\max_{x>0} Df_{Ro}(x) = Df_{Ro}(1), \quad \max_{x<0} Df_{Ro}(x) = Df_{Ro}(-1)$.
(f4) -1 and 1 are pre-periodic repelling points for f_{Ro}.
(f5) f_{Ro} has negative Schwarzian derivative: i.e., $1/\sqrt{|Df_{Ro}|}$ is convex on I^- and I^+.

We say that a map of the interval $f : I \to I$ is a *Rovella map* if it satisfies the properties (f1)–(f5) above. We denote this class of maps by $\mathcal{F}_R$. We refer to a Rovella map which is topologically conjugate to the doubling map as a *full Rovella map*.

3.2.3 *A Rovella Attractor is Partially Hyperbolic*

A compact invariant set $\Lambda \subset M$ is *partially hyperbolic* if the tangent bundle $T_\Lambda M$ splits into a continuous sum of sub-bundles $E \oplus F$, $D\hat{f}_s$-invariant, with E uniformly contracting, F contains the flow-direction $[\hat{f}]$ and there are $0 < \lambda < 1$ and $c > 0$ such that for all $s > 0$ and each $x \in \Lambda$

$$\left\| D\hat{f}_s \mid E^s_x \right\| \cdot \left\| D\hat{f}_{-s} \mid E^{cu}_{\hat{f}_s(x)} \right\| < c\,\lambda^s. \tag{13}$$

It follows from the construction and condition (3) on the eigenvalues at the origin, that a Rovella attractor is partially hyperbolic. In particular, besides the existence of the stable (uniformly contracting) foliation $\mathcal{F}^s$, there is a centre-unstable C^1 foliation $\mathcal{F}^{cu}$.

3.2.4 *Projection to the Interval*

Notation: From here on it will often be convenient to use the notation $\tilde{I} = \Sigma$ (recall $I = [-1/2, 1/2]$) and $\hat{I} = \mathcal{B}$, the domain of the Rovella flow constructed above.

The intersection of the foliations $\mathcal{F}^s$ and $\mathcal{F}^{cu}$ with the cross section $\tilde{I}$ induce a coordinate system (x, y) on $\tilde{I}$, i.e., any point in $\tilde{I}$ can be expressed as (x, y) where for all small ε, all points $(x+\varepsilon', y)$ for $|\varepsilon'|<\varepsilon$ are in the same unstable leaf as (x, y), and similarly $(x, y+\varepsilon')$ are in the same stable leaf as (x, y).

We define the map $\iota: \tilde{I} \to I$ as $\iota(x, y) = x$. Since our map $\tilde{f}$ preserves the stable foliation, (I, f) is a factor of $(\tilde{I}, \tilde{f})$. That is, $f \circ \iota = \iota \circ \tilde{f}$. To see this, let $(x, y) \in \tilde{I}$ and suppose that $(x', y') \in \tilde{I}$ is such that $\tilde{f}(x, y) = (x', y')$. From the definition of f, we have $x' = f(x)$. Then we compute

$$f \circ \iota(x, y) = f(x) = \iota(f(x), y') = \iota \circ \tilde{f}(x, y).$$

4 C^2 Cusp Maps

As in the previous section, given a Rovella map $\hat{f}$, if we take Poincaré sections twice then the study of the flow reduces to the study of one-dimensional maps. We will shortly define a wider class of one-dimensional maps which contains this class (and so the corresponding class of flows contains Rovella flows). First we give notation for one-sided derivatives of an interval map. If $A \subset \mathbb{R}$ and $f: A \to \mathbb{R}$ then for $x \in A$,

$$D^- f(x) := \lim_{y \nearrow x} \frac{f(x) - f(y)}{x - y} \quad \text{and} \quad D^+ f(x) := \lim_{y \searrow x} \frac{f(x) - f(y)}{x - y}$$

are the left and right derivatives at x respectively, provided the limits exist.

Definition $f: \bigcup_j I_j \to I$ is a *non-singular cusp map* if there exist constants $C, \alpha > 1$ and a finite collection $\{I_j\}_j$ of disjoint open subintervals of I such that

(i) for all $x, y \in \overline{I_j}$ we have $|Df_j(x) - Df_j(y)| < C|x-y|^\alpha$;
(ii) $D^+ f(a_j)$, $D^- f(b_j)$ exist and are equal to 0.

We denote the set of points a_j, b_j by Crit.

Dobbs [17] considered maps of this type, although he also allowed the maps to have some types of singularities at the boundaries of I_j. Note that the Lorenz-like maps considered in [16] are a subset of the cusp maps considered by Dobbs, but with extra expansion conditions.

Remark 2 Notice that if for some j, $b_j = a_{j+1}$, i.e., $I_j \cap I_{j+1}$ intersect, then f may not continuously extend to a well defined function at the intersection point b_j, since the definition above would then allow f to take either one or two values there. So in the definition above, the value of $f_j(a_j)$ is taken to be $\lim_{x \searrow a_j} f_j(x)$ and $f_j(b_j) = \lim_{x \nearrow b_j} f_j(x)$, so for each j, f_j is well defined on $\overline{I_j}$.

In this paper we will restrict to a particular subset of this class. We let $\mathcal{F}$ be the class of non-singular cusp maps with

(3) negative Schwarzian (i.e., $1/\sqrt{|Df|}$ is convex on each I_j).

This condition rules out the singularities considered by Dobbs. Moreover, it is clear that the class $\mathcal{F}_R$ of Rovella maps described in the previous section is included in $\mathcal{F}$. We let $\mathcal{F}_{ac} \subset \mathcal{F}$ denote the class of maps $f \in \mathcal{F}$ which have an acip μ_{ac} with positive Lyapunov exponent and which has density with respect to Lebesgue in L^p for some $p > 1$. Note that maps in $\hat{\mathcal{F}}_R$ as well as the non-singular maps in [16] are in $\hat{\mathcal{F}}_{ac}$. This can be derived for example from [15, Lemma 2.2] and the exponential decay shown in [16, 29].

Next we introduce the class of flows we shall deal with:

Definition The set $\hat{\mathcal{F}}$ is the class of flows on $\hat{I}$ which give rise to a Poincaré map on $\tilde{I}$ which is uniformly contracting in the vertical direction, the return time of $(x, y, 1)$ is of order $-\log|x|$ (as in (5)) and the map induced in the horizontal coordinate is in $\mathcal{F}$. The set $\hat{\mathcal{F}}_{ac}$ is defined similarly.

The study of the potential $\hat{\varphi}_t$ as in (1) for maps in $\hat{\mathcal{F}}$ reduces to the study of potentials φ_t as in (2). In order to prove the existence of equilibrium states for these potentials we need to further restrict our class to maps with good expansion properties. Note that our conditions are much weaker than those required for Rovella maps.

We define

$$\lambda_M = \lambda_M(f) := \sup\{\lambda(\mu) : \mu \in \mathcal{M}\}, \qquad \lambda_m = \lambda_m(f) := \inf\{\lambda(\mu) : \mu \in \mathcal{M}\}.$$

Then for $f \in \mathcal{F}$ we let

$$t^- := \inf\{t : p(t) > -\lambda_M t\} \quad \text{and} \quad t^+ := \sup\{t : p(t) > -\lambda_m t\}. \tag{14}$$

Remark 3 The arguments of [39] can be adapted to show that if $f \in \mathcal{F}$ then $\lambda_m \geq 0$. This implies that $t^+ > 0$. If $f \in \mathcal{F}_{ac}$ then by definition the acip has positive Lyapunov exponent. Therefore, as in [18, Theorem 3], see also [27, Theorem 3], μ_{ac} is an equilibrium state for $-\log|Df|$ and moreover $t^+ \geq 1$.

Since $\lambda_M \leq \sup_{x\in I} \log|Df(x)| < \infty$ for $f \in \mathcal{F}$, we also have $t^- < 0$. Note that if $t^- > -\infty$ then p is linear for all $t \leq t^-$. Similarly, if $t^+ < \infty$ then p is linear for all $t \geq t^+$.

The first theorem gives equilibrium states for our systems. In the context of multimodal maps this theory was first considered in [11], later extended for some cases by [37], and then for more general cases in [12, 13] and in complete generality in [24]. The following theorem is proved in the Appendix of [24]. We point out that this theorem was proved using a special class of inducing schemes which are compatible with our equilibrium states. One good property, which follows from negative Schwarzian derivative, is that these schemes also have bounded distortion. The expanding interval maps coming from the expanding Lorenz flow do not have negative Schwarzian derivative so new tools would have to be used to prove an analogous theorem in that case.

Theorem 1 *Let $f \in \mathcal{F}$. Then for all $t \in (t^-, t^+)$ there is a unique equilibrium state μ_t for φ_t. Moreover,*

(1) $h(\mu_t) > 0$;
(2) *the map $t \mapsto p(t)$ is C^1 in (t^-, t^+);*
(3) *if all $c \in$ Crit are not periodic or preperiodic then $t^- = -\infty$.*

We next consider the Lyapunov spectrum. Given $f \in \mathcal{F}$, we define the *pointwise Lyapunov exponent at $x \in I$* as

$$\lambda(x) := \lim_{n\to\infty} \frac{1}{n} \log|Df^n(x)|,$$

if the limit exists. For $\alpha \in \mathbb{R}$, we let

$$J(\alpha) := \left\{x \in I : \lambda(x) = \alpha\right\}$$

and

$$J' := \left\{ x \in I : \text{the limit } \lim_{n\to\infty} \frac{1}{n} \log|Df^n(x)| \text{ does not exist} \right\}.$$

The unit interval can be decomposed in the following way (the *multifractal decomposition*),

$$I = J' \cup \left(\bigcup_\alpha J(\alpha) \right).$$

As in Sect. 2, the function that encodes this decomposition is called the *multifractal spectrum of the Lyapunov exponents* and it is defined by

$$\mathfrak{L}_f(\alpha) := \dim_H(J(\alpha)).$$

This function was studied by Weiss [48] in the context of Axiom A maps.

As in the usual theory, if $p(t)$ is C^1 at $t \in \mathbb{R}$ and there exists an equilibrium state μ_t for $-t\log|Df|$ then $Dp(t) = -\lambda(\mu_t)$. Let

$$\alpha_1 := D^+p(t^-) \quad \text{and} \quad \alpha_2 := D^-p(t^+). \tag{15}$$

The following is proved as in [25].

Theorem 2 *Let $f \in \mathcal{F}_{ac}$. Then for all $\alpha \in (\alpha_1, \alpha_2)$, the Lyapunov spectrum satisfies the following relation*

$$\mathfrak{L}_f(\alpha) = \frac{1}{\alpha} \inf_{t\in\mathbb{R}} (p(t) + t\alpha) = \frac{1}{\alpha}(p(t_\alpha) + t_\alpha \alpha) = \frac{h(\mu_{t_\alpha})}{\alpha},$$

where t_α is such that $Dp(t_\alpha) = -\alpha$. Moreover, $\mathfrak{L}_f$ is C^1 in (α_1, α_2).

Remark 4 In the case of the quadratic Chebyshev map, $\alpha_1 = \alpha_2$, so the above theorem is empty in this case. This fact is shown via the conjugacy to the doubling map. Moreover, there are only two possible Lyapunov exponents, one corresponding to the repelling fixed point in ∂I and one corresponding to the acip. The former corresponds to a set of Hausdorff dimension 0 and the latter to a set of Hausdorff dimension 1. It can also be shown that $\alpha_1 = \alpha_2$ for a full Rovella map.

Remark 5 As in Remark 3, if $f \in \mathcal{F}_{ac}$ then $t^- < 0$ and $t^+ \geq 1$. Hence the interval (α_1, α_2) contains the interval $(\lambda(\mu_{ac}), \lambda(\mu_{max})]$ where μ_{ac} is the acip (the equilibrium state for $-t\log|Df|$ for $t = 1$) and μ_{max} is the measure of maximal entropy (this can be thought of as the equilibrium state for $-t\log|Df|$ for $t = 0$).

5 Thermodynamics of Flows

5.1 Thermodynamics of Suspension Flows

We will show that for certain natural potentials for the contracting Rovella flow, we can prove an equivalent of Theorem 1. Given the Rovella flow $\hat{f} = (\hat{f}_s)_{s\geq 0}$ on $\hat{I} \subset \mathbb{R}^3$, as shown Sect. 3, we can take a 2 dimensional Poincaré section $\tilde{I} \subset \mathbb{R}^3$ and get the first return map $\tilde{f} = \hat{f}_r : \tilde{I} \to \tilde{I}$ where r is the return time of a point in $\tilde{I}$ to $\tilde{I}$.

We can treat the Rovella flow as a semiflow over $\tilde{I}$. We will describe the abstract setup for semiflows. For more background on this general setup we refer to [3] which describes the simple relation between the semiflow and the map on the base which the semiflow is taken over. Much of what follows is very similar to thermodynamic formalism in the setting of Anosov flows, see [10, 14]. However, the singularity causes some difficulties, creating some non-uniform hyperbolicity. For some information on such systems, but principally for SRB measures, see [47]. For related recent work on the thermodynamics of semiflows over Countable Markov Shifts to prove our results see [7].

Suppose that $f : X \to X$ is a dynamical system. Let the *roof function* $\check{r} : X \to [0, \infty)$ be a continuous function and consider the space:

$$\check{X} := \{(x, s) \in X \times \mathbb{R} : 0 \le s \le \check{r}(x)\}/\sim,$$

where $(x, \check{r}(x)) \sim (f(x), 0)$ for every $x \in X$. The *suspension semiflow* $\check{f} = (\check{f}_s)_{s \ge 0}$ *over* f *with roof function* $\check{r}$ is defined as

$$\check{f}_s(x, u) := (x, u + s) \quad \text{if } u + s \in [0, \check{r}(x)].$$

The relevant class of measures here is

$$\mathcal{M}(f, \check{r}) := \left\{ \mu \in \mathcal{M}(f) : \int \check{r} \, d\mu < \infty \right\}.$$

It is shown in [3] that if μ is an f-invariant measure, possibly infinite, and $\int \check{r} \, d\mu < \infty$ then the product measure $\mu \times m$, where m is Lebesgue, is $\check{f}$-invariant. Indeed when $\check{r}$ is bounded away from zero there is a canonical identification between $\mathcal{M}(\check{f})$ and $\mathcal{M}(f, r)$: the map $\check{\imath} : \mathcal{M}(f, \check{r}) \to \mathcal{M}(\check{f})$ given by

$$\check{\imath}(\mu) := \frac{(\mu \times m)|_{\check{X}}}{(\mu \times m)(\check{X})} \tag{16}$$

is a bijection.

The entropy of a flow $(\check{X}, \check{f}, \check{\mu})$ can be defined by the metric entropy of the corresponding time 1 map. Abramov [1] proved that for semiflows, this is the same as

$$h(\check{f}, \check{\mu}) = \frac{h(f, \check{\mu} \circ \check{\imath}^{-1})}{\int r \, d(\check{\mu} \circ \check{\imath}^{-1})}. \tag{17}$$

We will take this definition.

Given a potential $\check{\varphi} : \check{X} \to \mathbb{R}$, we define the corresponding potential $\Delta_{\check{\varphi}} : X \to \mathbb{R}$ as

$$\Delta_{\check{\varphi}}(x) := \int_0^{\check{r}(x)} \check{\varphi}(x, s) \, dt.$$

By the identification of $\mathcal{M}(\check{f})$ and $\mathcal{M}(f, \check{r})$, coupled with the Abramov formula, we can write

$$P(\check{f}, \check{\varphi}) = \sup \left\{ \frac{1}{\int \check{r} \, d\mu} \left(h(f, \mu) + \int \Delta_{\check{\varphi}} \, d\mu \right) : \mu \in \mathcal{M}(f, \check{r}) \text{ and } - \int \Delta_{\check{\varphi}} \, d\mu < \infty \right\}.$$

Remark 6 If the underlying system (X, f) is a countable Markov shift, under certain smoothness conditions on the potential, a Variational Principle for the pressure was proved in [7]. We could extend that theory to the case of the Rovella flow with the potentials given below. However, since this isn't required to prove our results, we will not do the computations here.

We now return to the map $\tilde{f} : \tilde{I} \to \tilde{I}$. As in [3], there is a map between the suspension flow and the actual flow:

$$\check{p} : \check{I} = \left\{(x, s) \in \tilde{I} \times \mathbb{R} : 0 \le s \le \check{r}(x)\right\} / \sim \to \hat{I}.$$

We can define entropy of a measure $\hat{\mu} \in \mathcal{M}(\hat{f})$ as $h(\hat{\mu} \circ \check{p})$. Similarly we can define the pressure $P(\hat{f}, \hat{\varphi})$ as $P(\check{f}, \check{\varphi})$.

5.2 The Relation Between the one and Two Dimensional Systems

Later we will relate equilibrium states for f with those for $\tilde{f}$. This involves comparing the free energies of measures for f with those for $\tilde{f}$, so we will need a relation between $\mathcal{M}(f)$ and $\mathcal{M}(\tilde{f})$. We will use ideas from [5] to help with this. Note that in that paper the authors considered expanding rather than contracting Lorenz flows, but many of those ideas carry through to our contracting case. As in [5, Corollary 5.2], there is an injection from $\mathcal{M}(f)$ to $\mathcal{M}(\tilde{f})$. Moreover, given $\tilde{\mu} \in \mathcal{M}(\tilde{f})$ the measure $\tilde{\mu} \circ \iota^{-1}$ is clearly f-invariant. In the following lemma we show that in fact we have a bijection between $\mathcal{M}(f)$ and $\mathcal{M}(\tilde{f})$.

Lemma 1 *There is a bijection $\tilde{p} : \mathcal{M}(f) \to \mathcal{M}(\tilde{f})$. Moreover, $\iota \circ \tilde{p}$ is the identity on $\mathcal{M}(f)$ and $\tilde{p}$ and $\tilde{p}^{-1}$ take ergodic measures to ergodic measures.*

Proof Given $x \in I$, we let $\xi_x := \{(x, y) : y \in I\}$. This is a leaf of the $\mathcal{F}^s$ the stable foliation of $\tilde{I}$. For any potential $\tilde{\varphi} : \tilde{I} \to \mathbb{R}$, we define

$$\varphi_-(x) = \inf_{y \in \xi_x} \tilde{\varphi}(x, y) \quad \text{and} \quad \varphi_+(x) = \sup_{y \in \xi_x} \tilde{\varphi}(x, y).$$

Following [5, Sect. 5.1], we can show that if $\mu \in \mathcal{M}(f)$ then there is a unique measure $\tilde{\mu} \in \mathcal{M}(\tilde{f})$ such that for any continuous function $\tilde{\varphi} : \tilde{I} \to \mathbb{R}$, the limits

$$\lim_{n\to\infty} \int \left(\tilde{\varphi} \circ \tilde{f}^n\right)_- d\mu \quad \text{and} \quad \lim_{n\to\infty} \int \left(\tilde{\varphi} \circ \tilde{f}^n\right)_+ d\mu$$

exist, are equal, and coincide with $\int \tilde{\varphi}\, d\tilde{\mu}$. This determines the map $\tilde{p} : \mathcal{M}(f) \to \mathcal{M}(\tilde{f})$, which is injective. We will show that it is in fact a bijection between $\mathcal{M}(f)$ and $\mathcal{M}(\tilde{f})$.

The map ι gives us a natural way to get from $\mathcal{M}(\tilde{f})$ to $\mathcal{M}(f)$. The lemma will be proved if we can show that given $\tilde{\mu} \in \mathcal{M}(\tilde{f})$, for the measure $\nu := \tilde{\mu} \circ \iota^{-1} \in \mathcal{M}(f)$ we have $\tilde{\nu} = \tilde{\mu}$ (i.e. $\tilde{p} \circ \iota$ is the identity on $\mathcal{M}(\tilde{f})$).

As in [5, Corollary 5.2], we have, for a continuous $\tilde{\varphi} : \tilde{I} \to \mathbb{R}$ and $\tilde{\nu} := \tilde{p}(\nu)$,

$$\begin{aligned}\int \tilde{\varphi}\, d\tilde{\nu} &= \lim_{n\to\infty} \int \left(\tilde{\varphi} \circ \tilde{f}^n\right)_- d\nu = \lim_{n\to\infty} \int (\tilde{\varphi} \circ \tilde{f}^n)_-\, d(\tilde{\mu} \circ \iota^{-1}) \\ &= \lim_{n\to\infty} \int \left(\inf_{y' \in I} \tilde{\varphi} \circ \tilde{f}^n(x, y')\right) d\tilde{\mu}(x, y)\end{aligned}$$

since the integrand is independent of y. Because $\int \tilde{\varphi} \circ \tilde{f}^n \, d\tilde{\mu} = \int \tilde{\varphi} \, d\tilde{\mu}$, to complete the lemma it suffices to show that increasing n makes

$$\left| \int \left(\inf_{y' \in I} \tilde{\varphi} \circ \tilde{f}^n(x, y') \right) d\tilde{\mu}(x, y) - \int \tilde{\varphi} \circ \tilde{f}^n(x, y) \, d\tilde{\mu}(x, y) \right|$$

arbitrarily small (this follows similarly when we replace inf by sup). Since $\tilde{f}$ is uniformly contracting on each ξ_x by ($\star\star$), and $\tilde{\varphi}$ is uniformly continuous, for any $\varepsilon > 0$, for all large n,

$$\left| \left(\inf_{y' \in I} \tilde{\varphi} \circ \tilde{f}^n(x, y') \right) - \tilde{\varphi} \circ \tilde{f}^n(x, y) \right| < \varepsilon,$$

for all $(x, y) \in \tilde{I}$. Therefore,

$$\left| \int \left(\inf_{y' \in I} \tilde{\varphi} \circ \tilde{f}^n(x, y') \right) d\tilde{\mu}(x, y) - \int \tilde{\varphi} \circ \tilde{f}^n(x, y) \, d\tilde{\mu}(x, y) \right| < \varepsilon.$$

We can also replace inf with sup here. Hence $\tilde{\nu} = \tilde{\mu}$ as required.

Given an ergodic measure $\tilde{\mu} \in \mathcal{M}(\tilde{f})$, it is clear that $\tilde{p}^{-1}(\tilde{\mu}) = \iota^{-1}(\tilde{\mu})$ is ergodic. Conversely, given an ergodic measure $\mu \in \mathcal{M}(f)$, the ergodicity of $\tilde{p}(\mu)$ follows as in [5, Corollary 5.5]. $\square$

5.3 Relation Between Thermodynamics of Systems on the Interval, Square and Flow

For a potential $\varphi : I \to \mathbb{R}$, we define $\tilde{\varphi} : \tilde{I} \to \mathbb{R}$ to be $\tilde{\varphi}(x, y) = \varphi(x)$. Conversely, if $\tilde{\varphi} : \tilde{I} \to \mathbb{R}$ is a potential depending only on the first coordinate then we define $\tilde{\varphi}_1(x) := \tilde{\varphi}(x, y)$.

Lemma 2 *Given a potential $\varphi : I \to \mathbb{R}$, μ is an equilibrium state for (I, f, φ) if and only if $\tilde{\mu}$ is an equilibrium state for $(\tilde{I}, \tilde{f}, \tilde{\varphi})$.*

Proof Suppose that $\tilde{\mu}$ is an equilibrium state for $\tilde{\varphi}$. Then

$$h(\tilde{f}, \tilde{\mu}) + \int \tilde{\varphi} \, d\tilde{\mu} = P(\tilde{f}, \tilde{\varphi}).$$

We let μ be the projection of $\tilde{\mu}$ to I. It is easy to show that $\int \tilde{\varphi} \, d\tilde{\mu} = \int \tilde{\varphi}_1 \, d\mu$ and $h(\tilde{f}, \tilde{\mu}) = h(f, \mu)$. Then

$$h(f, \mu) + \int \tilde{\varphi}_1 \, d\mu = P(\tilde{f}, \tilde{\varphi}).$$

As in [10], $P(\tilde{f}, \tilde{\varphi}) = P(f, \tilde{\varphi}_1)$, so μ is an equilibrium state for $\tilde{\varphi}_1$.

To complete the proof of the lemma, we observe that the above computations also imply that if $\mu \in \mathcal{M}(f)$ is an equilibrium state for φ, then $\tilde{\mu} = \tilde{p}(\mu)$ is an equilibrium state for $\tilde{\varphi}$. $\square$

The following lemma gives us candidate equilibrium states for the Rovella flow.

Lemma 3 *For each $t \in (t^-, t^+)$ the equilibrium state μ_t for φ_t, projects to a measure $\tilde{\mu}_t \in \mathcal{M}(\tilde{f}, r)$.*

Proof The fact that μ_t projects to an invariant measure $\tilde{\mu}_t \in \mathcal{M}(\tilde{f})$ follows as in Lemma 2. It remains to show that $\int r\, d\tilde{\mu} < \infty$. By (5), $s(x) \asymp \log|x-c|$. By part (ii) of the definition of cusp maps, the integrability of $\log|Df|$ implies the integrability of r. The fact that $\log|Df| \in L^1(\mu_t)$ is clear from the definitions of $\mathcal{F}$ and μ_t. □

Next we extend to the flow. Any $\tilde{f}$-invariant measure $\tilde{\mu}$ on $\tilde{I}$ can be identified with a $\check{f}$-invariant measure $\check{\mu} = \check{\imath}(\tilde{\mu})$ where $\check{\imath}$ is defined in (16). Note by [5, Corollary 5.10] if $\nu \in \mathcal{M}(\tilde{f})$ is ergodic then $\check{\nu} \in \mathcal{M}(\check{f})$ is ergodic. We define the map

$$\hat{p} : \mathcal{M}(f) \xrightarrow{\tilde{p}} \mathcal{M}(\tilde{f}) \xrightarrow{\check{\imath}} \mathcal{M}(\check{f}) \xrightarrow{\check{p}} \mathcal{M}(\hat{f}).$$

Given $\mu \in \mathcal{M}(f)$, let $\hat{\mu} := \hat{p}(\mu)$.

Lemma 4 *Suppose that $\hat{\varphi} : \hat{I} \to \mathbb{R}$ gives a potential $\tilde{\varphi} = \Delta_{\hat{\varphi}} : \tilde{I} \to \mathbb{R}$ which depends only on the first coordinate. We set $\varphi(x) = \tilde{\varphi}(x, y)$ for any $y \in I$. Then μ is an equilibrium state for (I, f, φ) if and only if $\hat{\mu}$ is an equilibrium state for $(\hat{I}, \hat{f}, \hat{\varphi})$.*

Proof This follows as in [14, Theorem 4.4]: We may assume that $P(\varphi) = 0$. So

$$0 = P(\varphi) = h(\mu) + \int \varphi\, d\mu.$$

Then by Lemma 2 the measure $\tilde{\mu} \in \mathcal{M}(\tilde{f})$ has

$$0 = P(\tilde{\varphi}) = h(\tilde{\mu}) + \int \tilde{\varphi}\, d\tilde{\mu}.$$

We let $\hat{\mu} = \hat{p}(\tilde{\mu})$. Then by Lemma 3 and the Abramov formula,

$$h(\hat{\mu}) + \int \hat{\varphi}\, d\hat{\mu} = \frac{h(\tilde{\mu}) + \int \tilde{\varphi}\, d\tilde{\mu}}{\int r\, d\tilde{\mu}} = 0.$$

This computation also shows that $P(\hat{\varphi}) = 0$. Hence $\hat{\mu}$ is an equilibrium state for $\hat{\varphi}$. The converse argument follows similarly. □

Proposition 1 *For $\hat{\varphi}_t$ and φ_t as in (1) and (2) respectively, $\mu \in \mathcal{M}(f)$ is an equilibrium state for (I, f, φ_t) if and only if $\hat{\mu} = \hat{\imath}(\mu)$ is an equilibrium state for $(\hat{I}, \hat{f}, \hat{\varphi}_t)$.*

Proof Lemma 4 gives this immediately. □

Proof of Theorem A Proposition 1 added to Theorem 1 completes the proof. □

5.4 Lyapunov Spectrum for the Flow: Proof of Theorem B

To prove Theorem B, first note that given $x \in I$ with $\lambda(x) = \alpha$, all points $(x, y) \in \xi_x$ must lie in $K(\alpha)$. Moreover,

$$\left\{ \check{f}_s(x, y, 1) : y \in \xi_x \text{ and } s \in [0, r(x, y, z)) \right\} \subset K(\alpha).$$

Therefore, if we view $\hat{f}$ as a suspension flow over $(\tilde{I}, \tilde{f})$ with roof function r,

$$\dim_H(K(\alpha)) = \dim_H(K(\alpha) \cap \tilde{I}) + 1 = \dim_H(J(\alpha)) + 2 = L(\alpha) + 2.$$

Theorem 2 then gives L in terms of the pressure.

To complete the proof of Theorem B we need to check that the map from the suspension flow model to the flow $\hat{f}$ does not distort things too much; in particular is locally bilipschitz. This allows us to assert the first equality above. In [34] they refer to $(x, y, z) \in \hat{I}$ as a *regular point* if there exists $(\tilde{x}, \tilde{y}, \tilde{z}) \in \tilde{I}$ and a neighbourhood U_0 such that $\hat{f}_s : U_0 \to U_s$ is a diffeomorphism where U_s is a neighbourhood of $(\tilde{x}, \tilde{y}, \tilde{z})$. Note that any point in $\Lambda \cap \tilde{I}$ is regular.

For any regular point (x, y, z), there a neighbourhood of $(\tilde{x}, \tilde{y}, \tilde{z})$ in $\hat{I}$ such that the flow by $\hat{f}$ to the corresponding neighbourhood of (x, y, z) is conjugated by $\check{p}$ to the parallel flow on a cube. By the Tubular Flow Theorem of [34, Chap. 2], this conjugacy is bilipschitz in this neighbourhood. This proves Theorem B.

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